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CHARACTERIZATION OF $\{2(q+1)+2, 2; t, q\}$-MIN · HYPERS IN PG(t, q) $(t \geq 3, q \geq 5)$ AND ITS APPLICATIONS TO ERROR-CORRECTING CODES

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Let F be a set of f points in a finite projective geometry PG(t, q) of t dimensions (cf. Appendix I) where $t \geq 2$, $f \geq 1$ and q is a prime power. If (a) $|F \cap H| \geq m$ for any hyperplane H in PG(t, q) and (b) $|F \cap H| = m$ for some hyperplane H in PG(t, q), then F is said to be an $\{f, m; t, q\}$-min · hyper (or an $\{f, m; t, q\}$-minihyper) where $m \geq 0$ and $|A|$ denotes the number of elements in the set A.

Recently, all $\{2(q+1)+2, 2; 2, q\}$ – min · hypers in PG$(2, q)$ have been characterized by Hamada [10, 12] for any prime power $q \geq 3$. The purpose of this paper is to characterize all $\{2(q+1)+2, 2; t, q\}$-min · hypers in PG(t, q) for any integer $t \geq 3$ and any prime power $q \geq 5$ using the results in Hamada [6–11]. Using those results, all $(n, k, d; q)$-codes meeting the Griesmer bound (1.1) are characterized for the case $k \geq 3$, $d = q^{k-1} - (2 + 2q)$ and $q \geq 5$. Those results are a generalization of the results (due to Tamari) which have been published in Discrete Mathematics 49 (1984) 179–191.

1. Introduction

Let $V(n; q)$ be an n-dimensional vector space consisting of row vectors over a Galois field GF(q) of order q where n is a positive integer and q is a prime power. A k-dimensional subspace C of $V(n; q)$ is said to be an $(n, k, d; q)$-code (or a q-ary linear code with length n, dimension k, and minimum distance d) if the minimum (Hamming) distance of the code C is equal to d where $n > k \geq 3$ and $d \geq 1$ (cf. Blake and Mullin [2] and MacWilliams and Sloane [17]).

It is well known (cf. Griesmer [5] and Solomon and Stiffler [18]) that if there exists an $(n, k, d; q)$-code, then

$$n \geq \sum_{l=0}^{k-1} \left\lceil \frac{d}{q^l} \right\rceil \tag{1.1}$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$. The bound (1.1) (called the Griesmer bound) shows that in order to obtain an $(n, k, d; q)$-code C whose length n is minimum among $(*, k, d; q)$-codes for given integers k, d and q, it is sufficient to obtain an $(n, k, d; q)$-code meeting the Griesmer bound (1.1) in the case where there exists such a code for given integers k, d and q. Hence we shall consider the

Problem A.

(1) Find a necessary and sufficient condition for integers k, d and q that there exists an $(n, k, d; q)$-code meeting the Griesmer bound (1.1) for given integers k, d and q.

(2) Characterize all $(n, k, d; q)$-codes meeting the Griesmer bound (1.1) in the case where there exist such codes.

It is well known (cf. Baumert and McEliece [1] and Hamada and Tamari [14]) that for any given integers k and q, there exists some (large) integer d_0 (depending on k and q) such that there exists an $(n, k, d; q)$-code meeting the Griesmer bound (1.1) for any integer $d \geq d_0$. From the actual point of view, it is desirable to obtain a solution of Problem A for a comparatively small integer d in the case where there exists such a solution. Hence we shall confine ourself to the case $k \geq 3$ and $1 \leq d < q^{k-1}$ in this paper. In this case, d can be expressed uniquely as follows:

$$d = q^{k-1} - \sum_{\alpha=0}^{k-2} \varepsilon_\alpha q^\alpha \tag{1.2}$$

using some integers ε_α such that $0 \leq \varepsilon_\alpha \leq q-1$ and $(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{k-2}) \neq (0, 0, \ldots, 0)$. In the special case $1 \leq d \leq q^{k-1} - q$, d can be also expressed uniquely as follows:

$$d = q^{k-1} - \left(\varepsilon + \sum_{i=1}^{h} q^{\mu_i}\right) \tag{1.3}$$

using some odered set $(\varepsilon, \mu_1, \mu_2, \ldots, \mu_h)$ in $U(k-1, q)$ where $U(k-1, q)$ denotes a set of all ordered sets $(\varepsilon, \mu_1, \mu_2, \ldots, \mu_h)$ of integers ε, h and μ_i such that $0 \leq \varepsilon \leq q-1$, $1 \leq h \leq (k-2)(q-1)$, $1 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_h \leq k-2$ and $0 \leq n_l(\mu) \leq q-1$ for $l = 1, 2, \ldots, k-2$. Here $n_l(\mu)$ denotes the number of integers μ_i in $\mu \equiv (\mu_1, \mu_2, \ldots, \mu_h)$ such that $\mu_i = l$ for the given integer l. Note that in the case $d = q^{k-1} - (\varepsilon + \sum_{i=1}^{h} q^{\mu_i})$, the Griesmer bound (1.1) can be expressed as follows:

$$n \geq v_k - \left(\varepsilon + \sum_{i=1}^{h} v_{\mu_i+1}\right) \tag{1.4}$$

where $v_l = (q^l - 1)/(q-1)$ for any integer $l \geq 0$.

Problem A has been solved completely by Helleseth [16] and Tilborg [21] for the case $k \geq 3$, $q = 2$ and $1 \leq d < 2^{k-1}$. Hamada [6, 7] showed that in order to solve Problem A for the case $k \geq 3$ and $d = q^{k-1} - (\varepsilon + \sum_{i=1}^{h} q^{\mu_i})$, it is sufficient to characterize all $\{\sum_{i=1}^{h} v_{\mu_i+1} + \varepsilon, \sum_{i=1}^{h} v_{\mu_i}; k-1, q\}$-min·hypers in $PG(k-1, q)$ and solved Problem A completely for the case $k \geq 3$, $q \geq 3$, $\varepsilon \in \{0, 1\}$, $1 \leq h \leq k-2$ and $1 \leq \mu_1 < \mu_2 < \cdots < \mu_h \leq k-2$ using characterization of min·hypers in $PG(k-1, q)$ (cf. Theorems 2.1, 5.1 and 5.2 in [7] and Theorems 1.2, 2.8, 2.10 and 2.11 in [6]). (cf. Tamari [20] in the case $h = 1$).

Recently, all $\{2v_2+2, 2v_1; 2, q\}$-min · hypers (for the case $\varepsilon=2$, $h=2$ and $\mu_1=\mu_2=1$) have been characterized by Hamada [10, 12] for any prime power $q\geqslant 3$. The purpose of this paper is to characterize all $\{2v_2+2, 2v_1; t, q\}$-min · hypers for any integer $t\geqslant 3$ and any prime power $q\geqslant 5$ using the results in Hamada [6–11] (cf. Theorem 2.1 in Section 2). Using those results, all $(n, k, d; q)$-codes meeting the Griesmer bound (1.1) are characterized for the case $k\geqslant 3$, $d=q^{k-1}-(2+2q)$ and $q\geqslant 5$.

2. Main theorems

Let F be a set of f points in a finite projective geometry PG(t, q) of t dimensions where $t\geqslant 2$, $f\geqslant 1$ and q is a prime power. If (a) $|F\cap H|\geqslant m$ for any hyperplane H in PG(t, q) and (b) $|F\cap H|=m$ for some hyperplane H in PG(t, q), then F is said to be an $\{f, m; t, q\}$-min · hyper where $m\geqslant 0$ and $|A|$ denotes the number of elements in the set A (cf. Hamada and Tamari [13] and Hamada [6] with respect to a more general definition of a min · hyper).

Hamada [6, 7] showed that in the case $k\geqslant 3$ and $d=q^{k-1}-(\varepsilon+\sum_{i=1}^{h} q^{\mu_i})$, there is a one-to-one correspondence between a set of all $(n, k, d; q)$-codes meeting the Griesmer bound (1.4) and a set of all $\{\sum_{i=1}^{h} v_{\mu_i+1}+\varepsilon, \sum_{i=1}^{h} v_{\mu_i}; k-1, q\}$-min · hypers in PG$(k-1, q)$ if we introduce an equivalence relation between two $(n, k, d; q)$-codes as Definition 2.1 in [7]. Hence in order to solve Problem A for the case $k\geqslant 3$, $d=q^{k-1}-(\varepsilon+\sum_{i=1}^{h} q^{\mu_i})$ and $q\geqslant 3$, it is sufficient to solve the

Problem B.

(1) Find a necessary and sufficient condition for an ordered set $(\varepsilon, \mu_1, \mu_2, \ldots, \mu_h)$ in $U(t, q)$ that there exists a $\{\sum_{i=1}^{h} v_{\mu_i+1}+\varepsilon, \sum_{i=1}^{h} v_{\mu_i}; t, q\}$-min · hyper where $t\geqslant 2$ and $q\geqslant 3$.

(2) Characterize all $\{\sum_{i=1}^{h} v_{\mu_i+1}+\varepsilon, \sum_{i=1}^{h} v_{\mu_i}; t, q\}$-min · hypers in the case where there exist such min · hypers for a given ordered set $(\varepsilon, \mu_1, \mu_2, \ldots, \mu_h)$ in $U(t, q)$.

Problem B has been solved completely by Hamada [6–11] for any ordered set $(\varepsilon, \mu_1, \mu_2, \ldots, \mu_h)$ in $U(t, q)$ such that either (I) $\varepsilon\in\{0, 1\}$, $1\leqslant h\leqslant t-1$, $\mu_1<\mu_2<\cdots<\mu_h$ and $q\geqslant 3$ or (II) $\varepsilon\in\{0, 1\}$, $h=2$, $\mu_1=\mu_2=1$ and $q\geqslant 3$ or (III) $\varepsilon=0$, $h=2$, $\mu_1=\mu_2\geqslant 2$ and $q\geqslant 3$ or (IV) $\varepsilon=0$, $h=3$, $\mu_2=\mu_3=\mu_1+1$ and $q\geqslant 4$ or (V) $\varepsilon=2$, $h=1$, $\mu_1=1$ and $q\geqslant 3$ or (VI) $\varepsilon=0$, $h=3$, $\mu_1=\mu_2=\mu_3-1$ and $q\geqslant 5$ where $t\geqslant 2$ (cf. Appendix II and Lemmas 4.1 and 4.2 in Tamari [20]).

The purpose of this paper is to characterize all $\{2v_2+2, 2v_1; t, q\}$-min · hypers for any integer $t\geqslant 3$ and any prime power $q\geqslant 5$, i.e. to prove the following theorem.

Theorem 2.1. *Let t and q be any integer $\geqslant 3$ and any prime power $\geqslant 5$, respectively. Then F is a $\{2v_2+2, 2v_1; t, q\}$-min · hyper if and only if $F \in \mathfrak{F}(0, 0, 1, 1; t, q)$, i.e. F consists of two 0-flats and two 1-flats in $PG(t, q)$ which are mutually disjoint where $v_1 = 1$ and $v_2 = q+1$.*

Let C be a subspace of $V(f; q)$ generated by k row vectors of a $k \times f$ matrix G whose entries are elements of GF(q). If the rank of G over GF(q) is equal to $\tau(\leqslant k)$ and the maximum (Hamming) distance of C is equal to δ, then C is called a q-ary anticode with length f, dimension τ, and maximum distance δ and G is called a generator matrix of C. The concept of an anticode has been introduced by Farrell [4] in order to obtain an $(n, k, d; q)$-code meeting the Griesmer bound (1.1).

It is well known (cf. Ch. 17–§6 in MacWilliams and Sloane [17] and Example I in Appendix I) that in the case $k \geqslant 3$ and $d = q^{k-1} - (\varepsilon + \sum_{i=1}^{h} q^{\mu_i})$, there is a one-to-one correspondence between a set of all $(n, k, d; q)$-codes meeting the Griesmer bound (1.4) and a set of all q-ary anticodes, generated by a $k \times f$ matrix whose any two column vectors are linearly independent over GF(q), with length f and maximum distance $f - m$ if we introduce an equivalence relation between two codes as Definitions 2.1 and 5.1 in [7] where

$$f = \sum_{i=1}^{h} v_{\mu_i+1} + \varepsilon, \qquad m = \sum_{i=1}^{h} v_{\mu_i} \quad \text{and} \quad n = v_k - f. \tag{2.1}$$

Hence we have the following theorem from Theorem 2.1 and Theorem I in Appendix I.

Theorem 2.2. *Let k and q be any integer $\geqslant 3$ and any prime power $\geqslant 5$, respectively. In the case $d = q^{k-1} - (2+2q)$, C is an $(n, k, d; q)$-code meeting the Griesmer bound (1.1) if and only if C is congruent to some $(n, k, d; q)$-code constructed by using two 0-flats and two 1-flats in $PG(k-1, q)$ which are mutually disjoint.*

3. Propositions for the proof of Theorem 2.1

In order to prove Theorem 2.1, we shall prepare three propositions in this section.

Lemma 3.1. *Let F be any $\{2(q+1)+2, 2; t, q\}$-min · hyper where $t \geqslant 3$ and $q \geqslant 5$.*

(1) *If H is a hyperplane in $PG(t, q)$ such that $m(q+1) \leqslant |F \cap H| < (m+1)(q+1)$ for some integer m in $\{0, 1, 2\}$, then $F \cap H (\equiv F^*)$ is an $\{f, m; t, q\}$-min · hyper in H where $f = |F \cap H|$.*

(2) *There is no hyperplane H in $PG(t, q)$ such that $mq+4<|F\cap H|<(m+1)(q+1)$ for any integer m in $\{0, 1\}$.*

(3) *There exists at least one hyperplane H in $PG(t, q)$ such that $|F\cap H|\geq q+3$. In the special case $t\geq 4$, there exists a hyperplane H in $PG(t, q)$ such that $|F\cap H|\geq q+4$.*

(4) *For any hyperplane H in $PG(t, q)$ such that $|F\cap H|\geq q+3$, $|F\cap H| = q+3, q+4, 2q+2, 2q+3$ or $2q+4$.*

Proof.

(1) In the case $m=0$ (i.e. $0\leq |F\cap H|<q+1$), it follows from Proposition II.1 in Appendix II that F^* is an $\{f, 0; t, q\}$-min · hyper in H.

In the case $m\geq 1$, it follows from Proposition II.1 and $|F^*|<(m+1)(q+1)$ that if $|F^*\cap G|\geq m$ for any $(t-2)$-flat G in H, then there exists at least one $(t-2)$-flat G in H such that $|F^*\cap G|=m$, i.e. F^* is an $\{f, m; t, q\}$-min · hyper in H. Hence it is sufficient to show that $|F^*\cap G|\geq m$ for any $(t-2)$-flat G in H.

Suppose there exists a $(t-2)$-flat G in H such that $|F^*\cap G|\leq m-1$. Let H_i $(i=1, 2, \ldots, q)$ be q hyperplanes in PG(t, q), except for H, which contain G. Then $|F\cap(H_i\backslash G)| = |F\cap H_i| - |F\cap G|\geq 2-(m-1)$ for $i=1, 2, \ldots, q$. Hence $|F| = |F\cap H| + \sum_{i=1}^{q}|F\cap(H_i\backslash G)|\geq 3q+m>2q+4=|F|$, which is a contradiction. Hence (1) holds.

(2) Suppose there exists a hyperplane H in PG(t, q) such that $mq+4<|F\cap H|<(m+1)(q+1)$ for some integer m in $\{0, 1\}$. Then it follows from (1) that there exists a $(t-2)$-flat G in H such that $|F\cap G|=m$. Let H_i $(i=1, 2, \ldots, q)$ be q hyperplanes in PG(t, q), except for H, which contain G. Then $|F| = |F\cap H| + \sum_{i=1}^{q}\{|F\cap H_i| - |F\cap G|\}>2q+4=|F|$, which is a contradiction. Hence (2) holds.

(3) Since F is a $\{2(q+1)+2, 2; t, q\}$-min · hyper, there exists at least one hyperplane Π_0 in PG(t, q) such that $|F\cap\Pi_0|=2$, i.e. $F\cap\Pi_0$ consists of two points (denoted by P_1 and P_2) in PG(t, q). Let Δ be any $(t-2)$-flat in Π_0 such that $\{P_1, P_2\}\subset\Delta$ (i.e. $|F\cap\Delta|=2$) and let Π_i $(i=1, 2, \ldots, q)$ be q hyperplanes in PG(t, q), except for Π_0, which contain Δ.

Suppose $|F\cap\Pi_i|\leq 4$ for $i=1, 2, \ldots, q$. Then

$$|F| = |F\cap\Pi_0| + \sum_{i=1}^{q}\{|F\cap\Pi_i| - |F\cap\Delta|\}<2q+4=|F|,$$

which is a contradiction. Hence there exists at least one hyperplane Π in $\{\Pi_1, \Pi_2, \ldots, \Pi_q\}$ such that $|F\cap\Pi|\geq 5$. Since there is no hyperplane Π in PG(t, q) such that $4<|F\cap\Pi|<q+1$, this implies that $|F\cap\Pi|\geq q+1$.

In the case $|F\cap\Pi|\leq q+2$, i.e. $|F\cap\Pi|=(q+1)+\varepsilon$ for some integer ε in $\{0, 1\}$, it follows from (1) that $F\cap\Pi$ is a $\{v_2+\varepsilon, v_1; t, q\}$-min · hyper in Π where $v_1=1$ and $v_2=q+1$. Hence it follows from Proposition II.3 that $F\cap\Pi\in\mathfrak{F}_U(\varepsilon, 1; t, q)$, i.e. $F\cap\Pi$ consists of ε points and one 1-flat (denoted by V_1) in Π. Let G be any $(t-2)$-flat in Π which contain V_1 and let H_i $(i=1, 2, \ldots, q)$ be

q hyperplanes in PG(t, q), except for Π, which contain G. Since

$$|F| = |F \cap \Pi| + \sum_{i=1}^{q} |F \cap (H_i \backslash G)|, \qquad |F| = 2q+4 \quad \text{and} \quad |F \cap \Pi| \leq q+2,$$

there exists at least one hyperplane H in $\{H_1, H_2, \ldots, H_q\}$ such that $|F \cap (H \backslash G)| \geq 2$. This implies that $|F \cap H| = |F \cap G| + |F \cap (H \backslash G)| \geq q+3$. Hence there exists at least one hyperplane H in PG(t, q) such that $|F \cap H| \geq q+3$.

If there exists a hyperplane H in PG(t, q) such that $|F \cap H| = q+3$, it follows from (1) and Proposition II.6 that $F \cap H \in \mathfrak{F}(0, 0, 1; t, q)$, i.e. $F \cap H$ consists of two points (denoted by P_1 and P_2) and one 1-flat (denoted by W_1) in H.

In the case $t \geq 4$, there exists a $(t-2)$-flat G^* in H which contains a 1-flat W_1 and a point P_1. Let H_i^* $(i = 1, 2, \ldots, q)$ be q hyperplanes in PG(t, q), except for H, which contain G^*. Then it follows from $|F| = 2q+4$ and $|F \cap H| = q+3$ that there exists at least one hyperplane H^* in $\{H_1^*, H_2^*, \ldots, H_q^*\}$ such that $|F \cap (H^* \backslash G^*)| \geq 2$, i.e. $|F \cap H^*| = |F \cap G^*| + |F \cap (H^* \backslash G^*)| \geq q+4$. Hence (3) holds.

(4) It follows from (2) that (4) holds. This completes the proof. □

Definition 3.1. Let $(\varepsilon, \mu_1, \mu_2, \ldots, \mu_h)$ be any ordered set in $U(t, q)$ and let $\mathfrak{F}_U(\varepsilon, \mu_1, \mu_2, \ldots, \mu_h; t, q)$ denote a family of all sets $\bigcup_{i=0}^{h} V_i$ of a set $V_0 \equiv \{P_1, P_2, \ldots, P_\varepsilon\}$, a μ_1-flat V_1, a μ_2-flat $V_2, \ldots,$ a μ_h-flat V_h in PG(t, q) which are mutually disjoint where $P_1, P_2, \ldots, P_\varepsilon$ denote ε points in PG(t, q) and $V_0 = \emptyset$ in the case $\varepsilon = 0$. As occasion demands, we shall denote $\mathfrak{F}_U(\varepsilon, \mu_1, \mu_2, \ldots, \mu_h; t, q)$ by $\mathfrak{F}(\lambda_1, \lambda_2, \ldots, \lambda_\eta; t, q)$ where $\eta = h + \varepsilon$, $\lambda_i = 0$ $(i = 1, 2, \ldots, \varepsilon)$ and $\lambda_{\varepsilon+j} = \mu_j$ $(j = 1, 2, \ldots, h)$.

Remark 3.1. It is well known (cf. Theorems 2.2 and 2.3 in Hamada and Tamari [15] for example) that $\mathfrak{F}_U(\varepsilon, \mu_1, \mu_2, \ldots, \mu_h; t, q) \neq \emptyset$ if and only if either (a) $h = 1$ or (b) $h \geq 2$ and $\mu_{h-1} + \mu_h \leq t-1$ where $(\varepsilon, \mu_1, \mu_2, \ldots, \mu_h) \in U(t, q)$.

Proposition 3.1. *Let F be any $\{2(q+1)+2, 2; t, q\}$-min · hyper where $t \geq 3$ and $q \geq 5$. If there exists a hyperplane H in PG(t, q) such that $|F \cap H| = 2q+2$ or $2q+3$, then $F \in \mathfrak{F}(0, 0, 1, 1; t, q)$.*

Proof.

Case I: $|F \cap H| = 2q+2$. It follows from Lemma 3.1 that $F \cap H$ is a $\{2v_2, 2v_1; t, q\}$-min · hyper. Hence it follows from Proposition II.4 that $F \cap H \in \mathfrak{F}(1, 1; t, q)$. Since $|F| = 2q+4$ and $|F \cap H| = 2q+2$, this implies that $F \in \mathfrak{F}(0, 0, 1, 1; t, q)$.

Case II: $|F \cap H| = 2q+3$. It follows from Lemma 3.1 that $F \cap H$ is a $\{2v_2+1, 2v_1; t, q\}$-min · hyper. Hence it follows from Proposition II.5 that $F \cap H \in \mathfrak{F}(0, 1, 1; t, q)$. This implies that $F \in \mathfrak{F}(0, 0, 1, 1; t, q)$. □

Lemma 3.2. *Let F^* be any $\{q+4, 1; t, q\}$-min · hyper such that $F^* \subset H$ for some hyperplane H in $PG(t, q)$ and let R be any point in H such that $R \notin F^*$ where $t \geq 3$ and $q \geq 3$. Then there exists a $(t-2)$-flat G in H such that $R \in G$ and $|F^* \cap G| = 1$.*

Proof. It is well known that there are v_{t-1} 1-flats in the $(t-1)$-flat H which contain a point R. Let L_i $(i = 1, 2, \ldots, v_{t-1})$ be v_{t-1} 1-flats in H which contain R. Then $|F^*| = \sum_{i=1}^{\omega} |F^* \cap L_i|$ where $\omega = v_{t-1}$.

Case I: $t = 3$. Since $|F^*| = q + 4$, $v_2 = q + 1$ and $|F^* \cap L_i| \geq 1$ for $i = 1, 2, \ldots, q+1$, there exists at least one 1-flat G in $\{L_1, L_2, \ldots, L_{q+1}\}$ such that $|F^* \cap G| = 1$. Hence Lemma 3.2 holds in the case $t = 3$.

Case II: $t \geq 4$. Since $|F^*| = q + 4$ and $v_{t-1} > q + 4$ in the case $t \geq 4$, there exists at least one 1-flat Δ_1 in $\{L_1, L_2, \ldots, L_\omega\}$ such that $F^* \cap \Delta_1 = \emptyset$.

In the case $t = 4$, let G_i $(i = 1, 2, \ldots, q+1)$ be $q + 1$ 2-flats in H which contain Δ_1. Since $|F^*| = \sum_{i=1}^{q+1} |F^* \cap G_i|$ and $|F^* \cap G_i| \geq 1$ for $i = 1, 2, \ldots, q+1$, there exists at least one 2-flat G in $\{G_1, G_2, \ldots, G_{q+1}\}$ such that $|F^* \cap G| = 1$. Hence Lemma 3.2 holds in the case $t = 4$.

In the case $t \geq 5$, it can be shown that there exists a $(t-3)$-flat Δ_{t-3} in H such that $\Delta_{t-3} \supset \Delta_{t-4}$ and $F^* \cap \Delta_{t-3} = \emptyset$. Let G_i $(i = 1, 2, \ldots, q+1)$ be $q + 1/(t-2)$-flats in H which contain Δ_{t-3}. Then there exists at least one $(t-2)$-flat G in $\{G_1, G_2, \ldots, G_{q+1}\}$ such that $|F^* \cap G| = 1$. This completes the proof. □

Definition 3.2. Let V and W be a μ-flat and a ν-flat in PG(t, q), respectively, where $0 \leq \mu \leq \nu \leq t-1$. Let $V \oplus W$ denote the minimum flat in PG(t, q) which contains two flats V and W. In the special case $V \cap W = \emptyset$, $V \oplus W$ denotes a $(\mu + \nu + 1)$-flat in PG(t, q) which contains two flats V and W.

Lemma 3.3. *Let F be any $\{2(q+1)+2, 2; t, q\}$-min · hyper such that there exists a hyperplane H in $PG(t, q)$ such that $|F \cap H| = q + 4$ where $t \geq 3$ and $q \geq 5$. Let Q_1 and Q_2 be any two points in $F \backslash F^*$ and let R be a point in H such that $H \cap (Q_1 \oplus Q_2) = \{R\}$ where $F^* = F \cap H$. Then* (1) *$R \in F^*$ and* (2) *$F^* \backslash \{R\}$ is a $\{(q+1)+2, 1; t, q\}$-min · hyper in H.*

Proof. It follows from Lemma 3.1 that F^* is a $\{q+4, 1; t, q\}$-min · hyper in H.

(1) Suppose $R \notin F^*$. Then it follows from Lemma 3.2 that there exists a $(t-2)$-flat G in H such that $R \in G$ and $|F^* \cap G| = 1$. Let M_i $(i = 1, 2, \ldots, q)$ be q hyperplanes in PG(t, q), except for H, which contain G. Without loss of generality, we can assume that $Q_1 \in M_1$.

Since $R \in M_1$ and $Q_2 \in Q_1 \oplus R$, it follows that $Q_2 \in M_1$. This implies that $Q_1 \notin M_i$ and $Q_2 \notin M_i$ for $i = 2, 3, \ldots, q$. Since $|F \backslash (F^* \cup \{Q_1, Q_2\})| = q - 2$, there exists at least one hyperplane M in $\{M_2, M_3, \ldots, M_q\}$ such that $M \cap (F \backslash F^*) = \emptyset$. This implies that $|F \cap M| = |F^* \cap M| = |F^* \cap G| = 1$, which is a contradiction. Hence $R \in F^*$.

(2) Since F^* is a $\{q+4, 1; t, q\}$-min · hyper in H, $|F^* \cap G^*| \geq 1$ for any

$(t-2)$-flat G^* in H. Suppose there exists one $(t-2)$-flat G^* in H such that $R \in G^*$ and $|F^* \cap G^*| = 1$. Then it follows from the proof of (1) that there exists a hyperplane M in PG(t, q) such that $|F^* \cap M| = 1$, which is a contradiction. Hence $|(F^* \setminus \{R\}) \cap G^*| \geq 1$ for any $(t-2)$-flat G^* in H. Therefore, it follows from Proposition II.1 that $F^* \setminus \{R\}$ is a $\{q+3, 1; t, q\}$-min · hyper in H. □

Proposition 3.2. *Let F be any $\{2(q+1)+2, 2; t, q\}$-min · hyper where $t \geq 3$ and $q \geq 5$. If there exists a hyperplane H in PG(t, q) such that $|F \cap H| = q+4$, then $F \in \mathfrak{F}(0, 0, 1, 1; t, q)$.*

Proof. Let Q_i $(i = 1, 2, \ldots, q)$ denote q points in $F \setminus F^*$ and let $R_{\alpha\beta}$ $(1 \leq \alpha < \beta \leq q)$ denote a point in H such that $H \cap (Q_\alpha \oplus Q_\beta) = \{R_{\alpha\beta}\}$ where $F^* = F \cap H$. Then it follows from Lemma 3.3 that $R_{\alpha\beta} \in F^*$ and $F^* \setminus \{R_{\alpha\beta}\}$ is a $\{(q+1)+2, 1; t, q\}$-min · hyper in H. Hence it follows from Proposition II.6 that $F^* \setminus \{R_{\alpha\beta}\} \in \mathfrak{F}(0, 0, 1; t, q)$ for any integers α and β such that $1 \leq \alpha < \beta \leq q$. Since $F = (F^* \setminus \{R_{12}\}) \cup \{R_{12}, Q_1, Q_2, \ldots, Q_q\}$ and $F^* \setminus \{R_{12}\} \in \mathfrak{F}(0, 0, 1; t, q)$, it is sufficient to show that $\{R_{12}, Q_1, Q_2, \ldots, Q_q\}$ is a 1-flat in PG(t, q) in order to show that $F \in \mathfrak{F}(0, 0, 1, 1; t, q)$.

Suppose $\{R_{12}, Q_1, Q_2, \ldots, Q_q\}$ is not a 1-flat in PG(t, q). Then there exist three noncollinear points in $\{Q_1, Q_2, \ldots, Q_q\}$. Without loss of generality, we can assume that three points Q_1, Q_2 and Q_3 are noncollinear, i.e. R_{12}, R_{13} and R_{23} are all distinct. Furthermore, we can assume that R_{12}, R_{13}, R_{23} and R_{14} are all distinct.

Since $R_{\alpha\beta} \in F^*$ and $F^* \setminus \{R_{\alpha\beta}\} \in \mathfrak{F}(0, 0, 1; t, q)$ for any pair (α, β) in $\{(1,2), (1,3), (2,3)\}$, it follows that $F^* = V_1 \cup \{R_{12}, R_{13}, R_{23}\}$ where V_1 denotes a 1-flat in H. Since $R_{14} \in F^*$ and $R_{14} \notin \{R_{12}, R_{13}, R_{23}\}$, we have $R_{14} \in V_1$. This implies that $F^* \setminus \{R_{14}\} \notin \mathfrak{F}(0, 0, 1; t, q)$, which is a contradiction. Hence $\{R_{12}, Q_1, Q_2, \ldots, Q_q\}$ must be a 1-flat in PG(t, q). This completes the proof. □

Proposition 3.3. *Let F be any $\{2(q+1)+2, 2; 3, q\}$-min · hyper where $q \geq 5$. If there exists a 2-flat H in PG$(3, q)$ such that $|F \cap H| = q+3$, then $F \in \mathfrak{F}(0, 0, 1, 1; 3, q)$.*

Proof. Since $q \geq 5$, it follows from Lemma 3.1 and Proposition II.6 that $F \cap H \in \mathfrak{F}(0, 0, 1; 3, q)$, i.e. $F \cap H (\equiv F^*)$ consists of two points (denoted by P_1 and P_2) and one 1-flat (denoted by V_1) in H. Put $L = P_1 \oplus P_2$. Then L is a 1-flat in H and $L \cap V_1$ consists of one point (denoted by R) in V_1. Note that $F \cap L = \{P_1, P_2, R\}$, i.e. $|F \cap L| = 3$.

Let Q_i $(i = 1, 2, \ldots, q+1)$ denote $q+1$ points in $F \setminus F^*$ and let M_j $(j = 1, 2, \ldots, q)$ be q 2-flats in PG$(3, q)$, except for H, which contain L. Then there exists at least one 2-flat M in $\{M_1, M_2, \ldots, M_q\}$ such that $|M \cap \{Q_1, Q_2, \ldots, Q_{q+1}\}| \geq 2$. Since $\{P_1, P_2, R\} \subset F \cap M$, $V_1 \subset F$, $M \cap V_1 = \{R\}$ and

$|F| = 2q+4$, this implies that $5 \leq |F \cap M| \leq q+4$. Hence it follows from Lemma 3.1 that $q+1 \leq |F \cap M| \leq q+4$.

Case I: $|F \cap M| = q+4$. It follows from Proposition 3.2 that $F \in \mathfrak{F}(0, 0, 1, 1; 3, q)$.

Case II: $|F \cap M| = q+3$. It follows from Lemma 3.1 and Proposition II.6 that $F \cap M$ consists of two points (denoted by S_1 and S_2) and one 1-flat (denoted by W_1) in M where $V_1 \cap W_1 = \emptyset$ or $V_1 \cap W_1 = \{R\}$.

In the case $V_1 \cap W_1 = \{R\}$, it follows from $\{P_1, P_2\} \subset M$, $|W_1| = q+1$ and $|F \cap M| = q+3$ that $\{P_1, P_2\} = \{S_1, S_2\}$ and $F = V_1 \cup W_1 \cup \{P_1, P_2, Q\}$ where Q denotes a point in $\{Q_1, Q_2, \ldots, Q_{q+1}\}$ such that $Q \notin M$. Since $q \geq 5$ and $V_1 \cap W_1 = \{R\}$, there exists a 2-flat Π in PG$(3, q)$ such that $F \cap \Pi = \{R\}$, i.e. $|F \cap \Pi| = 1$, which is a contradiction. Hence $V_1 \cap W_1 = \emptyset$.

In the case $V_1 \cap W_1 = \emptyset$, it follows from $V_1 \subset F$, $W_1 \subset F$ and $|F| = 2(q+1)+2$ that $F \in \mathfrak{F}(0, 0, 1, 1; 3, q)$.

Case III: $|F \cap M| = q+2$. It follows from Lemma 3.1 and Proposition II.3 that $F \cap M \in \mathfrak{F}(0, 1; 3, q)$. On the other hand, it follows from $|F \cap L| = 3$, $L \subset M$ and $|F \cap M| = q+2$ that $F \cap M$ contains no 1-flat in M. This is a contradiction. Hence $|F \cap M| \neq q+2$.

Case IV: $|F \cap M| = q+1$. It follows from Lemma 3.1 and Proposition II.3 that $F \cap M \in \mathfrak{F}(1; 3, q)$. On the other hand, $F \cap M \notin \mathfrak{F}(1; 3, q)$ since $|F \cap M \cap L| = 3$. This is a contradiction. Hence $|F \cap M| \neq q+1$. This completes the proof. □

4. The proof of Theorem 2.1

(1) If $F \in \mathfrak{F}(0, 0, 1, 1; t, q)$, i.e. $F \in \mathfrak{F}_U(2, 1, 1; t, q)$, it follows from Proposition II.2 that F is a $\{2v_2+2, 2v_1; t, q\}$-min · hyper where $v_1 = 1$ and $v_2 = q+1$.

(2) Conversely, let F be any $\{2(q+1)+2, 2; t, q\}$-min · hyper. Then it follows from Lemma 3.1 that there exists some hyperplane H in PG(t, q) such that $|F \cap H| = q+3$, $q+4$, $2q+2$, $2q+3$ or $2q+4$.

Case I: $|F \cap H| = q+3$ and $t = 3$. It follows from Proposition 3.3 that $F \in \mathfrak{F}(0, 0, 1, 1; t, q)$.

Case II: $|F \cap H| = q+3$ and $t \geq 4$. It follows from Lemma 3.1 that there exists at least one hyperplane Π in PG(t, q) such that $|F \cap \Pi| \geq q+4$.

Case III: $|F \cap H| = q+4$. It follows from Proposition 3.2 that $F \in \mathfrak{F}(0, 0, 1, 1; t, q)$.

Case IV: $|F \cap H| = 2q+2$ or $2q+3$. It follows from Proposition 3.1 that $F \in \mathfrak{F}(0, 0, 1, 1; t, q)$.

Case V: $|F \cap H| = 2q+4$. since $F = F \cap H$, F is a $\{2q+4, 2; t, q\}$-min · hyper in the $(t-1)$-flat H. We shall prove "$F \in \mathfrak{F}(0, 0, 1, 1; t, q)$" by induction on t.

In the case $t = 3$, it follows from Proposition II.7 and Lemma 3.1 that there is no $\{2q+4, 2; 3, q\}$-min · hyper F in the 2-flat H. This implies that there is no hyperplane H in PG$(3, q)$ such that $|F \cap H| = 2q + 4$ in the case $t = 3$. Hence it follows from Cases I–IV that $F \in \mathfrak{F}(0, 0, 1, 1; t, q)$ in the case $t = 3$.

In the case $t \geq 4$, it follows from induction on t that $F \cap H$ consists of two 0-flats and two 1-flats in H which are mutually disjoint. Since $F = F \cap H$, this implies that $F \in \mathfrak{F}(0, 0, 1, 1; t, q)$. This completes the proof. □

Appendix I. A connection between a min · hyper and an anticode

A finite projective geometry PG(t, q) of t dimensions may be defined as a set of points satisfying the following conditions:

(1) A point in PG(t, q) is represented by (v) where v is a nonzero element of a Galois field GF(q^{t+1}).

(2) Two points (v_1) and (v_2) represent the same point when and only when there exists a nonzero element σ of GF(q) such that $v_2 = \sigma v_1$.

(3) A μ-flat in PG(t, q) is defined as a set, $\{(a_0 v_0 + a_1 v_1 + \cdots + a_\mu v_\mu) \mid \cdots\}$, of $v_{\mu+1}$ points in PG(t, q) where $0 \leq \mu < t$ and a_i's run independently over the elements of GF(q) and are not all simultaneously zero and (v_0), $(v_1), \ldots, (v_\mu)$ are linearly independent over the coefficient field GF(q), in other words, they do not lie on a $(\mu - 1)$-flat. In the special case $\mu = 0$, 1 or $t - 1$, a 0-flat, a 1-flat and a $(t-1)$-flat in PG(t, q) are also called a point, a line and a hyperplane, respectively. A (-1)-flat denotes the empty set $\emptyset$.

Since every nonzero element of GF(q^{t+1}) may be represented as a nonzero vector in $W(t+1; q)$, every point in PG(t, q) may be represented by $(\underset{\sim}{c})$ using some nonzero vector $\underset{\sim}{c}$ in $W(t+1; q)$ (cf. Appendix I in Hamada [6] in detail) where $W(k; q)$ denotes a k-dimensional vector space over GF(q) consisting of column vectors and $(\underset{\sim}{c}_1) = (\underset{\sim}{c}_2)$ when and only when there exists some nonzero element σ of GF(q) such that $\underset{\sim}{c}_2 = \sigma \underset{\sim}{c}_1$. Between a min · hyper and an anticode (defined in Section 2), there is the following connection.

Theorem I (Hamada [7]). *Let k and q be any integer ≥ 3 and any prime power, respectively, and let f and m be some integers such that $0 \leq m < f \leq v_k$. Let $\underset{\sim}{e}_l$ $(l = 1, 2, \ldots, f)$ be f nonzero vectors in $W(k; q)$ such that any two vectors in $(\underset{\sim}{e}_1, \underset{\sim}{e}_2, \ldots, \underset{\sim}{e}_f)$ are linearly independent. Then $\{(\underset{\sim}{e}_1), (\underset{\sim}{e}_2), \ldots, (\underset{\sim}{e}_f)\}$ is an $\{f, m; k-1, q\}$-min · hyper in PG$(k-1, q)$ if and only if $[\underset{\sim}{e}_1 \underset{\sim}{e}_2 \cdots \underset{\sim}{e}_f]$ is a $k \times f$ generator matrix of a q-ary anticode with length f and maximum distance $f - m$.*

In order to show a connection between a min · hyper and an $(n, k, d; q)$-code meeting the Griesmer bound (1.1), we shall give the following example (cf. Example 5.1 in [7] in detail).

Example I. Consider the case $k = 3$, $d = 4$ and $q = 3$. In this case, $h = 1$, $\varepsilon = 2$, $\mu_1 = 1$ and $v_3 = (3^3 - 1)/(3 - 1) = 13$. Let $\underset{\sim}{c}_i$ $(i = 1, 2, \ldots, 13)$ be 13 vectors given by

c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9	c_{10}	c_{11}	c_{12}	c_{13}
0	0	0	0	1	1	1	1	1	1	1	1	1
0	1	1	1	0	0	0	1	1	1	2	2	2
1	0	1	2	0	1	2	0	1	2	0	1	2

Let $F=\{(c_1),(c_3),(c_5),(c_6),(c_7),(c_{10})\}$, $G=[c_1c_3c_5c_6c_7c_{10}]$ and $G^*=[c_2c_4c_8c_9c_{11}c_{12}c_{13}]$. Let C be a subspace in $V(6;3)$ generated by 3 row vectors of G and let C^* be a subspace in $V(7;3)$ generated by 3 row vectors of G^*. Then F is a $\{6,1;2,3\}$-min · hyper such that $F\in\mathfrak{F}(0,0,1;2,3)$ (i.e. F is a set of a 1-flat $\{(c_1),(c_5),(c_6),(c_7)\}$ and two 0-flats (c_3) and (c_{10}) in PG(2, 3) which are mutually disjoint) and C is a 3-ary anticode with length 6 and maximum distance 5 and C^* is a $(7,3,4;3)$-code meeting the Griesmer bound (1.1). In this case, C^* is said to be a $(7,3,4;3)$-code constructed by using a 1-flat $\{(c_1),(c_5),(c_6),(c_7)\}$ and two 0-flats (c_3) and (c_{10}) in PG(2, 3).

Appendix II. Preliminary results for the proof of Theorem 2.1

We shall describe several results, due to Hamada [6–12], which are used in proving Theorem 2.1. (Cf. Tamari [19] with respect to (1) in Proposition II.1)

Proposition II.1 (Hamada [7]). *Let t and q be any integer ≥ 2 and any prime power, respectively, and let $v_l=(q^l-1)/(q-1)$ for any integer $l\geq 0$.*

(1) *Let σ_i's be any integers such that $0\leq\sigma_1\leq q$ and $0\leq\sigma_i\leq q-1$ $(2\leq i\leq t-1)$. If $m\geq\sum_{\alpha=1}^{t-1}\sigma_\alpha v_\alpha$, then $f\geq\sum_{\alpha=1}^{t-1}\sigma_\alpha v_{\alpha+1}$ for any $\{f,m;t,q\}$-min · hyper F.*

(2) *Let ε_α's be any integers such that $0\leq\varepsilon_0\leq q$, $0\leq\varepsilon_\alpha\leq q-1$ $(\alpha=1,2,\ldots,t-1)$ and $(\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_{t-1})\neq(0,0,\ldots,0)$. If F is a set of $\sum_{\alpha=0}^{t-1}\varepsilon_\alpha v_{\alpha+1}$ points in PG(t,q) such that $|F\cap H|\geq\sum_{\alpha=1}^{t-1}\varepsilon_\alpha v_\alpha$ for any hyperplane H in PG(t,q), then F is a $\{\sum_{\alpha=0}^{t-1}\varepsilon_\alpha v_{\alpha+1},\sum_{\alpha=1}^{t-1}\varepsilon_\alpha v_\alpha;t,q\}$-min · hyper.*

(3) *Let V be any θ-flat in PG(t,q) where $t\geq 3$ and $2\leq\theta\leq t-1$. If F is a set of $\sum_{\alpha=0}^{t-1}\varepsilon_\alpha v_{\alpha+1}$ points in V such that $|F\cap\Pi|\geq\sum_{\alpha=1}^{t-1}\varepsilon_\alpha v_\alpha$ for any $(\theta-1)$-flat Π in V, then F is a $\{\sum_{\alpha=0}^{t-1}\varepsilon_\alpha v_{\alpha+1},\sum_{\alpha=1}^{t-1}\varepsilon_\alpha v_\alpha;t,q\}$-min · hyper in V.*

Proposition II.2 (Hamada [7]). *Let $(\varepsilon,\mu_1,\mu_2,\ldots,\mu_h)$ be any ordered set in $U(t,q)$ such that either* (I) *$h=1$ or* (II) *$h\geq 2$ and $\mu_{h-1}+\mu_h\leq t-1$ where $t\geq 2$ and $q\geq 2$. Then $\mathfrak{F}_U(\varepsilon,\mu_1,\mu_2,\ldots,\mu_h;t,q)\neq\emptyset$. If $F\in\mathfrak{F}_U(\varepsilon,\mu_1,\mu_2,\ldots,\mu_h;t,q)$, then F is a $\{\sum_{i=1}^{h}v_{\mu_i+1}+\varepsilon,\ \sum_{i=1}^{h}v_{\mu_i};t,q\}$-min · hyper.*

Proposition II.3 (Hamada [6]). *Let $(\varepsilon,\mu_1,\mu_2,\ldots,\mu_h)$ be any ordered set in $U(t,q)$ such that $\varepsilon\in\{0,1\}$, $1\leq h\leq t-1$ and $1\leq\mu_1<\mu_2<\cdots<\mu_h\leq t-1$ where $t\geq 2$ and $q\geq 3$.*

(1) *In the case* $h = 1$, *F is a* $\{v_{\mu_1+1} + \varepsilon, \quad v_{\mu_1};\ t, q\}$-*min · hyper if and only if* $F \in \mathfrak{F}_U(\varepsilon, \mu_1; t, q)$.
(2) *In the case* $h \geq 2$ *and* $\mu_{h-1} + \mu_h \leq t - 1$, *F is a* $\{\sum_{i=1}^{h} v_{\mu_i+1} + \varepsilon, \sum_{i=1}^{h} v_{\mu_i};\ t, q\}$-*min · hyper if and only if* $F \in \mathfrak{F}_U(\varepsilon, \mu_1, \mu_2, \ldots, \mu_h; t, q)$.
(3) *In the case* $h \geq 2$ *and* $\mu_{h-1} + \mu_h \geq t$, *there is no* $\{\sum_{i=1}^{h} v_{\mu_i+1} + \varepsilon, \sum_{i=1}^{h} v_{\mu_i};\ t, q\}$-*min · hyper.*

Proposition II.4 (Hamada [8, 9]). *Let t and* μ *be any integers such that* $1 \leq \mu < t$ *and let q be any prime power* ≥ 3.
(1) *In the case* $t \geq 2\mu + 1$, *F is a* $\{2v_{\mu+1}, 2v_{\mu}; t, q\}$-*min · hyper if and only if* $F \in \mathfrak{F}(\mu, \mu; t, q)$.
(2) *In the case* $t \leq 2\mu$, *there is no* $\{2v_{\mu+1}, 2v_{\mu}; t, q\}$-*min · hyper.*

Proposition II.5 (Hamada [8, 9]). *Let t and* μ *be any integers such that* $1 \leq \mu < t$ *and let q be any prime power* ≥ 4.
(1) *In the case* $t \geq 2\mu + 1$, *F is a* $\{2v_{\mu+1} + v_{\mu}, 2v_{\mu} + v_{\mu-1}; t, q\}$-*min · hyper if and only if* $F \in \mathfrak{F}(\mu - 1, \mu, \mu; t, q)$.
(2) *In the case* $t \leq 2\mu$, *there is no* $\{2v_{\mu+1} + v_{\mu}, 2v_{\mu} + v_{\mu-1}; t, q\}$-*min · hyper.*

Proposition II.6 (Hamada [10, 11]). *Let t and* μ *be any integers such that* $1 \leq \mu < t$ *and let q be any prime power* ≥ 5.
(1) *In the case* $t \geq 2\mu$, *F is a* $\{v_{\mu+1} + 2v_{\mu}, v_{\mu} + 2v_{\mu-1}; t, q\}$-*min-hyper if and only if* $F \in \mathfrak{F}(\mu - 1, \mu - 1, \mu; t, q)$.
(2) *In the case* $t < 2\mu$, *there is no* $\{v_{\mu+1} + 2v_{\mu}, v_{\mu} + 2v_{\mu-1}; t, q\}$-*min · hyper.*

Proposition II.7 (Hamada [10, 12]).
(1) *In the case* $q \geq 5$, *there is no* $\{2v_2 + 2, 2v_1; 2, q\}$-*min · hyper where* $v_1 = 1$ *and* $v_2 = q + 1$.
(2) *In the case* $q = 3$, *F is a* $\{2v_2 + 2, 2v_1; 2, 3\}$-*min · hyper if and only if there exist some noncollinear points* (v_0), (v_1) *and* (v_2) *in* $PG(2, 3)$ *such that* $F = \Omega \backslash \{(v_0), (v_1), (v_2)\}$ *where* $v_1 = 1$, $v_2 = 4$ *and* Ω *denotes a set of all points in* $PG(2, 3)$.
(3) *In the case* $q = 4$, *F is a* $\{2v_2 + 2, 2v_1; 2, 4\}$-*min · hyper if and only if there exist some noncollinear points* (v_0), (v_1) *and* (v_2) *in* $PG(2, 4)$ *such that either*
   (a) $F = L_0 \cup L_1 \cup \{(c_0 v_0 + v_1 + v_2), \quad (c_1 v_0 + \alpha v_1 + v_2), \quad (c_2 v_0 + \alpha^2 v_1 + v_2)\}$ *for some elements* c_0, c_1 *and* c_2 *in* $\{0, 1, \alpha, \alpha^2\}$ *or*
   (b) $F = L_0 \cup \{(v_2), (v_1 + v_2), (c v_0 + v_1 + v_2), (c v_0 + \alpha v_1 + v_2), (c\alpha v_0 + \alpha v_1 + v_2), (c v_0 + \alpha^2 v_1 + v_2), (c\alpha^2 v_0 + \alpha^2 v_1 + v_2)\}$ *for some element c in* $\{1, \alpha, \alpha^2\}$ *or*
   (c) $F = (L_0 \backslash \{(v_1)\}) \cup (L_1 \backslash \{(v_2)\}) \cup (M_2 \backslash \{(c v_1 + v_2)\}) \cup \{(c\alpha v_1 + v_2), (c\alpha^2 v_1 + v_2)\}$ *for some element c in* $\{1, \alpha, \alpha^2\}$ *where* $v_1 = 1$, $v_2 = 5$,

$L_0 = (v_0) \oplus (v_1)$, $L_1 = (v_0) \oplus (v_2)$, $M_2 = (v_0) \oplus (cv_1 + v_2)$ and α is a primitive element of $GF(2^2)$ such that $\alpha^2 = \alpha + 1$ and $\alpha^3 = 1$.

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